New Bounds for Probability of Error of Coded and Uncoded TQAM in AWGN Channel

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Abstract We investigate the performance of coded modulation scheme based on the application of integer codes to triangular quadrature amplitude modulation (TQAM). An upper and a lower bounds for symbol error probability (SER) in the case of AWGN channel are derived. These bounds are so closed that it makes the calculation of the exact value of SER unnecessary in practice.

1 Introduction

The term coded modulation means a combination of a scheme of coding and modulation techniques. Nowadays, in modern digital communication systems, high-order modulation is preferred for high-speed data transmission. One of the most popular modulation in commercial communication systems is square quadrature amplitude modulation (SQAM). SQAM scheme with its simple detection procedure is easy for implementation and demonstrates a good performance.

Recently, the triangular quadrature amplitude modulation (TQAM) was proposed. In TQAM constellation the signal points are vertexes of a lattice of equilateral triangles and the constellation is symmetric with respect to the origin. The comparison of TQAM with SQAM given in [7] shows that the former is more power efficient while preserves the low detection complexity of the latter. In [8] a general formula for calculating the average energy per symbol of the TQAM is derived and symbol error rate (SER) and bit error rate (BER) of the TQAM in the presence of additive white Gaussian noise (AWGN) is analyzed.

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Codes over finite rings and their applications in coding theory have been investigated by many researchers during the past several decades. It is well known that algebraic theory of block codes over finite rings have severe problems with coding for multidimensional constellations. The reason is, that in two or higher dimensions the Hamming distance is inappropriate. One possible solution of that problem is given in [1], where Huber introduced Mannheim distance and has applied codes over Gaussian integers for that distance. The weakness of that construction comes from the fact that based on a given code over Gaussian integers we arrange the points in the constellation.

So called integer codes, codes defined over finite rings of integers, are more useful from practical point of view. In contrast to the traditional block codes they are designed to correct errors of a given type. It means that for a given channel and modulator we can construct integer code capable of correcting the type of errors, which are the most common for this channel.

In this work we shall investigate the performance of coded modulation scheme based on the application of integer codes to TQAM constellation. Necessary definitions and results are given in Sect. 2. In Sect. 3 an upper and a lower bound for symbol error probability (SER) in the case of AWGN channel are derived. These bounds are so closed that it makes the calculation of the exact value of SER unnecessary in practice. The error performance of integer codes in case of TQAM constellation over AWGN channel is discussed in Sect. 4. Section 5 describes the simulations that have been carried out.

2 Preliminaries

2.1 TQAM Constellation

In this paper we will consider TQAM constellation of $M = 2^{2m}$ signal points placed in $L = 2^m$ rows parallel to real axis with L signal point in each row. The points form a lattice of equilateral triangles and the constellation is symmetric with respect to the origin. The power gain of M-ary TQAM over M-ary SQAM in decibels [8] is

$$10 \log_{10} \left(\frac{8M-8}{7M-4} \right) \xrightarrow[M\to\infty]{} 0.5799 \text{ dB}$$

For $M = 16,\ 64,\ 256$ the power gain is 0.458, 0.5505 and 0.5726, respectively.

An example of 16-ary TQAM is given in Fig. 1. The correspondence between signal points and all 16 sets of four bits presented in the figure is referred to as *Grey mapping*. In the case of SQAM this mapping guarantees that the 4-bit sets corresponding to the closest points differ only in one bit. In the case of TQAM it is impossible to realize such a mapping since the number of closest points can be up to six. But the same property holds for two (of three possible) direction in the

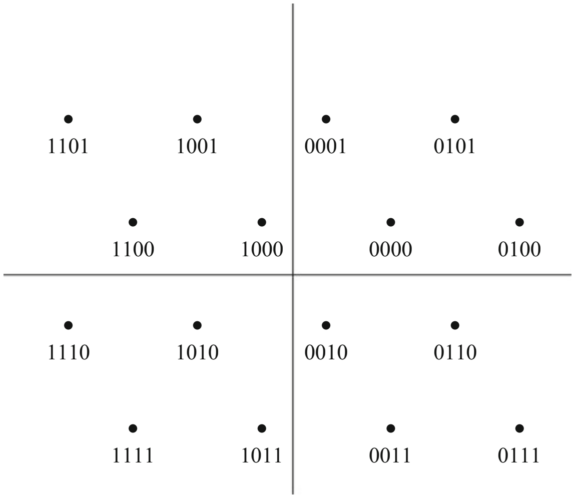


Fig. 1 16-TQAM constellation

constellation. Examples of 16-ary and 64-ary TQAMs with Grey mapping are given in [8], too.

Remark Some authors use (in the case of SQAM) the term Grey codding, but we avoid intentionally this notation. A coded modulation with Grey mapping decreases BER but it does not include decoding procedure and does not affect SER. We refer this case to as *uncoded case with Grey mapping* in contrast to the case of using integer codes.

2.2 *Integer Codes*

Herein we give some necessary definitions and notations which we shall use in the next sections. More details the reader can find in the mentioned papers.

Integer codes were proposed by Varshamov and Tenengolz [9] in 1965 for correcting single insertion/deletion per codeword, but in [3, 4] it was demonstrated that such classes of codes are very suitable for realization of coded modulation procedures. The applications of integer codes with different modulation schemes, in partial with SQAM, are discussed also in [5, 6].

Definition 2.1 [4] Let $\mathbf{Z}_A$ be the ring of integers modulo A. An *integer code* of length n with parity-check matrix $H \in \mathbf{Z}_A^{m\times n}$, is referred to as a subset of $\mathbf{Z}_A^n$, defined by

$$\mathbf{C}(H, \mathbf{d}) = \{\mathbf{c} \in \mathbf{Z}_A^n \mid \mathbf{c}H^T = \mathbf{d} \bmod A\}$$

where $\mathbf{d} \in \mathbf{Z}_A^m$.

Assume that a signal point s_i of a given signal constellation is sent through an AWGN-channel. At the other end the detector estimates the received signal r_i and gives signal point s_j at the output. If $j \neq i$ the detector has taken a wrong decision. In terms of block codes over $\mathbf{Z}_A$ the aforesaid can be described in the following way. When a codeword $\mathbf{c} \in \mathbf{C}(\mathbf{w}, d)$ is sent through a communication channel (usually noisy) the received vector can be written in the form

$$\mathbf{r} = \mathbf{c} + \mathbf{e},$$

where $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbf{Z}_A^n$ denotes the error vector. It is clear that the different signal points have not the same chance to be a result of decision process. The probability signal point s_j to appear at the output of the detector depends on the Euclidean distance between s_j and really-sent signal point s_i. In terms of codes over $\mathbf{Z}_A$ it means that the elements of $\mathbf{Z}_A$ are not equally probable as a value taken by e_i. Which elements of $\mathbf{Z}_A$ are more probable depends on the chosen indexing of the signal points by the elements of $\mathbf{Z}_A$. Therefore, it makes sense to consider (there is a point in considering) the next definition.

Definition 2.2 [4] The code $\mathbf{C}(H, \mathbf{d})$ is a t-multiple $(\pm e_1, \pm e_2, \ldots, \pm e_s)$-error correctable if it can correct up to t errors with values from the set $\{\pm e_1, \pm e_2, \ldots, \pm e_s\}$ which are occurred in a codeword.

Remark Without loss of generality in the definitions above we can assume that $\mathbf{d} = \mathbf{0}$. For convenience of a notation we shall use $\mathbf{C}$ instead of $\mathbf{C}(H, \mathbf{0})$.

To decode integer codes one can use a hard decoding algorithm [6]. This algorithm uses a look-up table which maps each syndrome value to the corresponding error vector. So, the complexity of the algorithm is linear with respect to the alphabet size A.

3 Bounds for SER and BER

Suppose that a signal point $\mathbf{s}$ is sent through a communication channel. At the other end the detector estimates the received signal $\mathbf{r}$ and has to take a decision: which signal point has been sent. Let the channel be an AWGN channel with power spectral density N_0 watts/hertz. Then

$$\mathbf{r} = \mathbf{s} + \mathbf{n},$$

where $\mathbf{n}$ is two dimensional zero-mean Gaussian random process with variance $\sigma^2 = N_0/2$.

Hence, at any arbitrary time the value of $\mathbf{n}(x, y)$ is statistically characterized by the Gaussian probability density function,

$$p(x, y) = \frac{1}{2\pi\sigma^2}\, e^{-\frac{x^2+y^2}{2\sigma^2}}.$$

Let q_u denote the probability of right decision of the detector, that is the probability of correct detection per symbol. In the case of uncoded SQAM constellation the detector takes the right decision if the received signal $\mathbf{r}$ belongs to a square with center the sent signal point $\mathbf{s}$ and side equals $2d$, which is the minimal possible distance between two signal points. In this case is more conveniently to consider the noise as $\mathbf{n} = (n_x, n_y)$ where n_x and n_y are independent (orthogonal) zero-mean Gaussian random processes with power spectral density $N_0/2$ (variance $\sigma^2 = N_0/2$). The probability of correct demodulation in the SQAM case is widely treated in the literature and well known (e.g., [2]). In [4] the reader can find also the probability of correct decoding in the case of L^2-SQAM coded with integer codes.

Unfortunately, in the case of TQAM the detection region are hexagon with side equals $a = 2d/\sqrt{3}$ (the regions of contour points are more complex) and the consideration of the noise as a set of two orthogonal and independent noises does not help so much as in the square case.

In order to avoid the complex calculations we suggest a different approach. Our idea is to approximate the detection regions by circles with center the sent signal point. At least for hexagonal region the inscribed and circumscribed circles give very tight lower and upper bounds for the probability of correct detection (upper and lower bounds for the probability of error, respectively). For the contour points circles are also a good approximation because of the part of the region outside the circle has relatively small contribution to the probability.

If $\mathbf{D}$ is the detection region then

$$q_u = \Pr(\mathbf{r} \in \mathbf{D}) = \iint_D \frac{1}{2\pi\sigma^2}\, e^{-\frac{x^2+y^2}{N_0}}\, dxdy.$$

In the case when $\mathbf{D}$ is a circle with radius λd we get

$$q_u = 1 - e^{-\frac{\lambda^2 d^2}{N_0}}$$

As we mentioned above q_u for L^2-SQAM is well known and it is given by the formula

$$q_u = \frac{1}{L^2}[1 + (L-1)\mathrm{erf}(\gamma)]^2,$$

where $\gamma = \frac{d}{\sqrt{N_0}} = \sqrt{\frac{3}{2(L^2-1)} \cdot \frac{E_s}{N_0}}$, $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$, and E_s is the average energy per signal point, that is, $\mathrm{SNR}_s = E_s/N_0$ is the signal-to-noise ratio per symbol.

This enable us to test our idea. The inscribed and circumscribed circles of the square with side $2d$ have radius d and $d\sqrt{2}$, respectively. Therefore we have to expect that

$$1 - e^{-\frac{d^2}{N_0}} < q_u < 1 - e^{-\frac{2d^2}{N_0}}$$

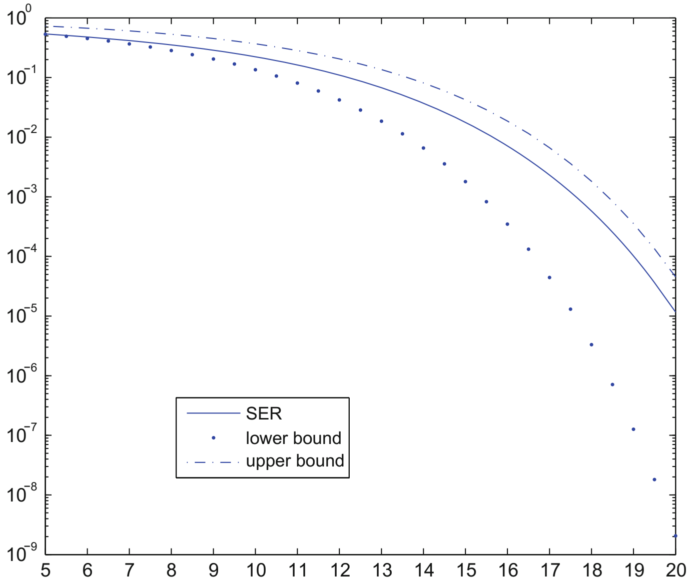


Fig. 2 Probability of symbol error for uncoded 16-SQAM

or for the symbol error (SER) p_u:

$$e^{-\frac{2d^2}{N_0}} < p_u < e^{-\frac{d^2}{N_0}}.$$

In the case of 16-SQAM (i.e., $L = 4$) the inequality for p_u as a function of SNR per symbol has the form

$$e^{-\frac{\mathrm{SNR}_s}{5}} < p_u < e^{-\frac{\mathrm{SNR}_s}{10}}$$

ant it is graphically represented in Fig. 2.

In the case of L^2-TQAM the distance between adjacent symbols is also $2d$, but the distance between horizontal rows is $h = d\sqrt{3}$ (which is the altitude of the of equilateral triangles). Then [8]

$$\frac{d^2}{N_0} = \frac{12}{7L^2 - 4} \cdot \frac{E_s}{N_0} = \frac{12}{7L^2 - 4} \cdot \mathrm{SNR}_s. \tag{1}$$

The radius of the inscribed circle for the hexagon is d, and the one of the circumscribed circle is $a = 2d/\sqrt{3}$ (see Fig. 3). Therefore we have to expect that

$$1 - e^{-\frac{d^2}{N_0}} < q_u < 1 - e^{-\frac{4d^2}{3N_0}}$$

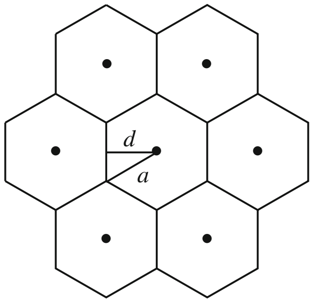


Fig. 3 Inscribed and circumscribed circles' radius

or for the symbol error (SER) p_u:

$$e^{-\frac{4d^2}{3N_0}} < p_u < e^{-\frac{d^2}{N_0}}. \tag{2}$$

Replacing d^2/N_0 by (1) we get

$$e^{-\frac{16}{7L^2-4}\mathrm{SNR}_s} < p_u < e^{-\frac{12}{7L^2-4}\mathrm{SNR}_s}. \tag{3}$$

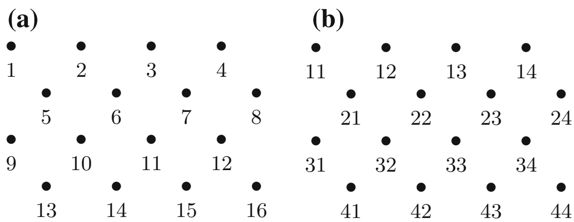
4 Integer Codes and TQAM

Herein we describe applications of integer codes to 16-TQAM. One possibility is to number the signal points by $\mathbb{Z}_{17}\backslash\{0\} = \{1, 2, \ldots, 16\}$ and to use the integer code with $\mathbf{H} = (1, 2)$ over the ring $\mathbb{Z}_{17}$ of integers modulo 17. This code is a single $(\pm 1, \pm 3, \pm 4, \pm 5)$-error correcting code and can correct a wrong detection of the sent point as one of the six neighbor (adjacent) points (see Fig. 4a).

Another possibility is the use of two codes over $\mathbb{Z}_5$ with check matrix $\mathbf{H} = (1, 2)$ and numbering the points by pair of integers as it is shown in Fig. 4b.

Following the idea given in the previous section one can decide to estimate probability of symbol error by a larger detection region that includes the adjacent (at distance $2d$) points of the sent signal point. The radii of the circles are $2a = 4d/\sqrt{3}$

Fig. 4 Coded 16-TQAM



and $5a/2 = 5d/\sqrt{3}$, respectively. Therefore the expectation for the probability of symbol error P_c is bounded by

$$e^{-\frac{100}{7L^2-4}\mathrm{SNR}_s} < P_c < e^{-\frac{64}{7L^2-4}\mathrm{SNR}_s}. \quad (4)$$

In partial, for 16-TQAM we have

$$e^{-\frac{25}{27}\mathrm{SNR}_s} < P_c < e^{-\frac{16}{27}\mathrm{SNR}_s}. \quad (5)$$

But such a way of reasoning is wrong because it does not take in account the parameters of the code—the length of the codeword and how many errors per a codeword it can correct. Hence one have to bear in mind that one or more symbols will be incorrect decoded if the number of error detection per a codeword is larger than the code capacity. For example, if two successive signal points are received outside the circle with radius d they cannot be correctly decoded by a single error correcting code of length two (as the code used for simulations). Thus a symbol error appears. The probability for this even is a square of the error probability in uncoded case. Hence the probability for symbol error P_c in coded case has to satisfy the inequality

$$e^{-\frac{32}{7L^2-4}\mathrm{SNR}_s} < P_c < e^{-\frac{24}{7L^2-4}\mathrm{SNR}_s}. \quad (6)$$

In partial for 16-TQAM we have:

$$e^{-\frac{8}{27}\mathrm{SNR}_s} < P_c < e^{-\frac{2}{9}\mathrm{SNR}_s}. \quad (7)$$

5 Simulations

To compare the obtained bounds with real performance we have developed a software that simulates communication based on the 2^k-TQAM in AWGN channel for $k = 4, 6, 8$.

The software generalizes a random sequence of bits, transforms it into a sequence of signal points and adds to them AWGN corresponding to the prescribed (input) SNR. Then realizes the procedure of detection (and decoding if necessary) and compare the obtained points with the sent ones to calculate the SER. Also, the software transforms the sequence of detected points into a sequence of bits and calculates the BER. Three modes of simulation are realized:

- ‘St’—uncoded case without any fixed rules of mapping k-bits sets to points;
- ‘grey’—uncoded case with Grey mapping (Fig. 1 for $k = 4$);
- ‘intc’—coded with a integer code (for $k = 4$ is used mapping given in Fig. 4b).

We carried out the simulations with pseudo-random sequences of length $\approx 10^9$ bits. As it was expected, the SERs for both uncoded cases, non-Grey (mode ‘St’) and

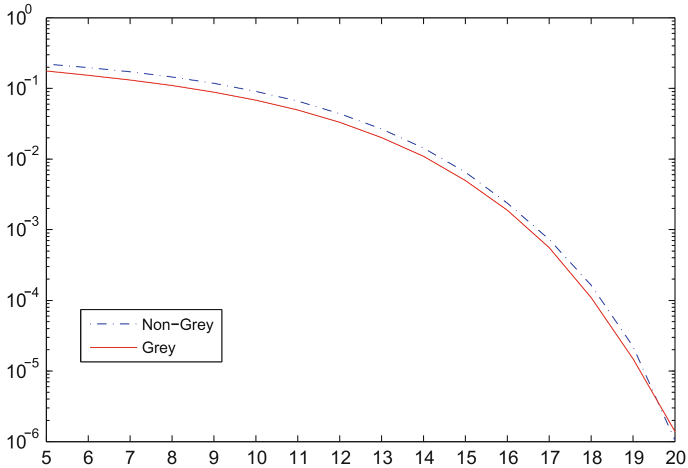


Fig. 5 BER of uncoded 16-TQAM—Grey and non-Grey mapping

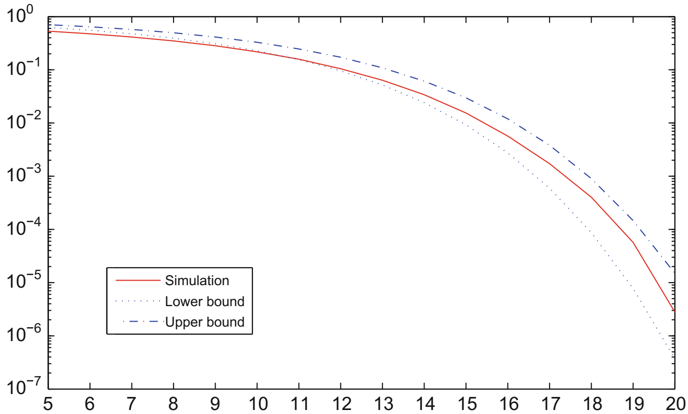


Fig. 6 SER of uncoded 16-TQAM—simulation result

Grey mapping, are almost the same, but the Grey mapping gives smaller BER. The bit error rates for uncoded 16-TQAM with and without Grey mapping are given in Fig. 5.

Figure 6 presents the comparison of the obtained by simulation SER with lower and upper bounds for uncoded case given by inequality (3). For small values of SNR all three graphs are very closed, the simulation even gives better result, but for the values that of practical interest the simulation graph is between the others as inequality (3) requires. The explanation is that for small SNR the deviation from (3) generated by contour points of the constellation is larger and it reflects on the mean SER.

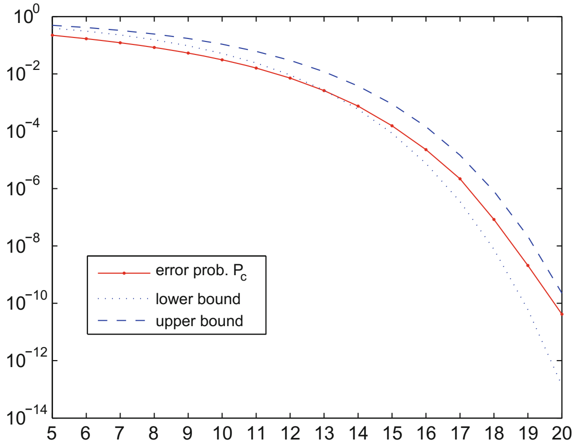


Fig. 7 SER of coded 16-TQAM—simulation result

Figure 7 demonstrates the result of simulation of 16-TQAM coded with single error integer code with $\mathbf{H} = (1, 2)$. The simulation confirms the inequality (7).

6 Conclusion

We derive a lower and an upper bounds for symbol error probability for TQAM. Our considerations and simulations show that this bounds give an acceptable for practice estimation of symbol error probability both in uncoded and coded with integer codes cases. The graphs show that in the all cases the real performance follows the behavior of the upper bound—its graph is just shifted in the better direction graph of the upper bound.

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